Fundamental Problems in Statistical Relational AI

Tutorial - KR 2024

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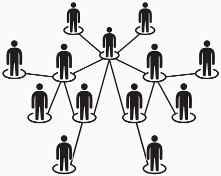
StaRAI

What are Relational Domains?

Social Networks



What are Relational Domains?

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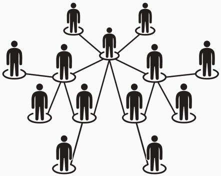
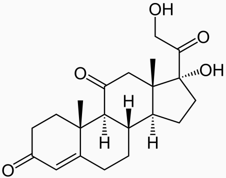


Molecules



What are Relational Domains?

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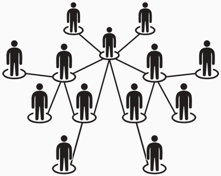

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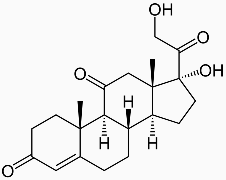
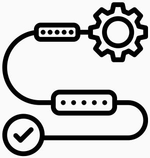
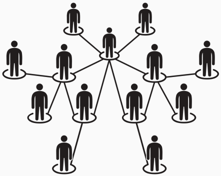


Business Processes



Modeling Relational Domains in First Order Logic

Social Networks



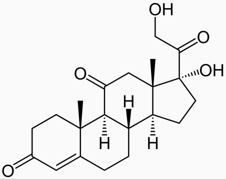
$\texttt{Freinds}(x, y)$

$\texttt{Smokes}(x)$

$\texttt{Cancer}(x)$

$\vdots$

Molecules



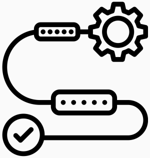
$\texttt{Bond}(x, y)$

$\texttt{Carbon}(x)$

$\texttt{Oxygen}(x)$

$\vdots$

Business Processes



$\texttt{Teaches}(x, y)$

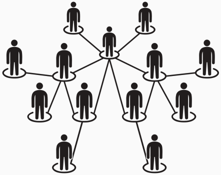
$\texttt{Professor}(x)$

$\texttt{Student}(x)$

$\vdots$

Modeling Knowledge in First Order Logic

Social Networks

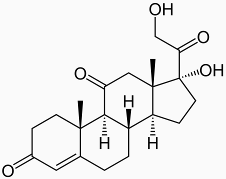


$\forall xy.\mathtt{Fr}(x,y) \rightarrow \mathtt{Fr}(y,x)$

$\forall xy.\mathtt{Sm}(x) \wedge \mathtt{Fr}(x,y) \rightarrow \mathtt{Sm}(y)$

$\vdots$

Molecules

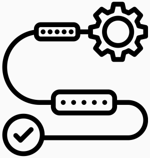


$\forall x.\mathtt{H}(x) \rightarrow \exists y.\mathtt{Bond}(x,y)$

$\forall x.\mathtt{O}(x) \rightarrow \exists^{\leq 2} y.\mathtt{Bond}(x,y)$

$\vdots$

Business Processes



$\forall xy.\mathtt{Prof}(x) \wedge \mathtt{Tch}(x,y) \rightarrow \mathtt{Stud}(y)$

$\vdots$

Modelling Relational Domains in First Order Logic

PROBLEMS![1]

- **Laziness**: Too much work to list out all rules!

[1]Russell and Norvig. Artificial Intelligence: A Modern Approach

Modelling Relational Domains in First Order Logic

PROBLEMS![1]

- **Laziness**: Too much work to list out all rules!
- **Theoretical Ignorance**: We don't have all the rules!

[1] Russell and Norvig. Artificial Intelligence: A Modern Approach

Modelling Relational Domains in First Order Logic

PROBLEMS![1]

- **Laziness**: Too much work to list out all rules!
- **Theoretical Ignorance**: We don't have all the rules!
- **Practical Ignorance**: Maybe there are no rules — inherent stochasticity!

[1] Russell and Norvig. Artificial Intelligence: A Modern Approach

Statistical Relational AI

Statistical Relational AI

=

Logic + Probability

Ingredients of Statistical Relational Learning

SRL ingredients:

- A set of Herbrand models $\Omega^{(n)}$ in function-free First Order Logic
- A parametric probability distribution $\mathbb{P}_\theta : \Omega \to [0, 1]$

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The Learning Problem:

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Fundamental Problems in SRL

"What are the fundamental problems in Statistical Relational Learning?"

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The Learning Problem:

- $\theta^* = \mathrm{argmax}_\theta \, \mathbb{P}_\theta(\omega^*)$ **No Consistent Estimation**

Tractability

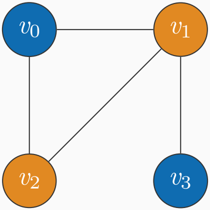
Markov Logic[2]: An Example

A contact-tracing model:

$$w_1 : \texttt{Covid}(\texttt{x})$$
$$w_2 : \texttt{Covid}(\texttt{x}) \wedge \texttt{Contact}(\texttt{x}, \texttt{y}) \rightarrow \texttt{Covid}(\texttt{y})$$

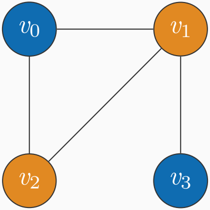
[2] Richardson and Domingos. Markov Logic Networks. 2006

MLNs: An Example



- $\texttt{Covid(x)}$
  $n_1(\omega) = |\{v_i : \omega \models \texttt{Covid}(\texttt{v}_\texttt{i})\}|$

MLNs — An Example



- $\mathtt{Covid}(\mathtt{x}) \land \mathtt{Contact}(\mathtt{x}, \mathtt{y}) \to \mathtt{Covid}(\mathtt{y})$
  $n_2(\omega) = |\{(v_i, v_j) : \omega \models \phi_2(\mathtt{v_i}, \mathtt{v_j})\}|$

Markov Logic Networks: Almost Formally

- Weighted quantifier-free first-order logic formulas:

$$\{w_i : \phi_i\}$$

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$$\texttt{Pr}(\omega) := \frac{1}{Z} \exp\left(\sum_i w_i n_i(\omega)\right) \tag{1}$$

Markov Logic Networks: Almost Formally

- Weighted quantifier-free first-order logic formulas:

$$\{w_i : \phi_i\}$$

- Probability distribution on all finite structures of size n:

$$\mathtt{Pr}(\omega) := \frac{1}{Z} \exp\left(\sum_i w_i n_i(\omega)\right) \tag{1}$$

- The partition function — main source of intractability:

$$Z := \sum_{\omega \models \Phi_\infty} \exp\left(\sum_i w_i n_i(\omega)\right) \tag{2}$$

Weighted First Order Model Counting

Symmetric Weighted First Order Model Counting (WFOMC):

$$\text{WFOMC}(\Phi, n) := \sum_{\omega \models \Phi} \mathrm{w}(\omega)$$

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$$\mathsf{WFOMC}(\Phi, n) := \sum_{\omega \models \Phi} \mathrm{w}(\omega)$$

$$\mathrm{w}(\omega) = \prod_{\substack{\omega \models g \\ g \in \mathcal{G}}} w(pred(g)) \prod_{\substack{\omega \models \neg g \\ g \in \mathcal{G}}} \bar{w}(pred(g)).$$

Weighted First Order Model Counting

Symmetric Weighted First Order Model Counting (WFOMC):

$$\mathsf{WFOMC}(\Phi, n) := \sum_{\omega \models \Phi} \mathrm{w}(\omega)$$

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WFOMC is **Tractable** if it can be computed in polynomial time w.r.t the **domain size (n)**.

WFOMC: An Example

$$\forall \mathsf{x}.\mathsf{KR}(\mathsf{x}) \rightarrow \mathtt{Intelligent}(\mathsf{x})$$

$$w(\mathsf{KR}) = 2.5 \qquad \bar{w}(\mathsf{KR}) = 0.5$$

$$w(\mathtt{Intelligent}) = 0.5 \qquad \bar{w}(\mathtt{Intelligent}) = 1.5$$

KR(a)	Intelligent(a)	w(ω)
1	1	$2.5 \times 0.5 = 1.25$
1	0	$2.5 \times 1.5 = 1.25$
0	1	$5 \times 0.5 = 1.5$
0	0	$5 \times 1.5 = 7.5$

$$\text{WFOMC}(\Phi, 1) = 1.25 + 1.5 + 7.5$$

WFOMC $\equiv$ Partition Function

$$Z := \sum_{\omega \models \Phi_\infty} \exp\left(\sum_i w_i n_i(\omega)\right)$$

$$\Phi_\infty \wedge \bigwedge_i \forall FV[\phi_i].\big(R_i(FV[\phi_i]) \leftrightarrow \phi_i\big) \tag{3}$$

$$w(R_i) = \exp(w_i) \qquad \bar{w}(R_i) = 1$$
$$w(*) = 1 \qquad \bar{w}(*) = 1$$

WFOMC $\equiv$ Partition Function: An Example

$$w : \mathsf{KR}(\mathsf{x}) \rightarrow \mathtt{Intelligent}(\mathsf{x})$$

WFOMC $\equiv$ Partition Function: An Example

$$w : \mathsf{KR}(\mathsf{x}) \rightarrow \mathtt{Intelligent}(\mathsf{x})$$

WFOMC encoding for the partition function:

$$\forall \mathsf{x}.\mathsf{R}(\mathsf{x}) \leftrightarrow \left(\mathsf{KR}(\mathsf{x}) \rightarrow \mathtt{Intelligent}(\mathsf{x})\right)$$

$$w(R) = \exp(w) \qquad w(*) = 1 \qquad \bar{w}(*) = 1$$

How Intractable (practically)?

WFOMC is a #P complete problem in general[3].

[3] Beame, Van den Broeck, Gribkoff, Suciu. PODS 2015.

How Intractable (practically)?

WFOMC is a #P complete problem in general[3].

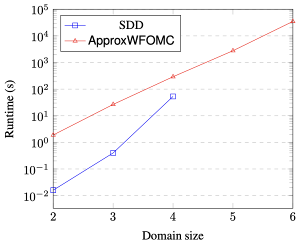
But is WMC not practically quite scalable?

[3] Beame, Van den Broeck, Gribkoff, Suciu. PODS 2015.

How Intractable?

$$w_1 : \texttt{Stress}(\texttt{x}) \rightarrow \texttt{Smokes}(\texttt{x})$$
$$w_2 : \texttt{Smokes}(\texttt{x}) \land \texttt{Fr}(\texttt{x}, \texttt{y}) \rightarrow \texttt{Smokes}(\texttt{y})$$
$$w_3 : \texttt{Fr}(\texttt{x}, \texttt{y}) \land \texttt{Fr}(\texttt{y}, \texttt{z}) \rightarrow \texttt{Fr}(\texttt{x}, \texttt{z})$$

HOW INTRACTABLE[4]?



(a) Runtime of various WFOMC methods for the `transitive-smokers-mln` problem for various domain sizes.

Figure 1: 10000 seconds is more than 2 hours

[4] Bremen and Kuzelka. IJCAI 2020

WFOMC: Fundamental Problems

What fragments of first-order logic admit tractable WFOMC?

[5] Beame, Van den Broeck, Gribkoff and Suciu. PODS 2015.
[6] den Broeck et al. KR 2014

WFOMC: Fundamental Problems

What fragments of first-order logic admit tractable WFOMC?

There is an intractable FO^3 formula[5]

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WFOMC: Fundamental Problems

What fragments of first-order logic admit tractable WFOMC?

There is an intractable FO^3 formula[5]

FO^2 is tractable[6]

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FO^2 Language: 1-Types

We have a language with **at most two variables** , with the following predicates:

- A unary predicate $\mathsf{KR}(x)$
- A binary predicate $\mathtt{Shaves}(x, y)$

FO^2 Language: 1-Types

We have a language with **at most two variables** , with the following predicates:

- A unary predicate $\mathsf{KR}(x)$
- A binary predicate $\mathsf{Shaves}(x, y)$

We have the following set of unary properties also called 1-types:

$$\neg\mathsf{KR}(c) \land \neg\mathsf{Shaves}(c, c)$$

$$\neg\mathsf{KR}(c) \land \mathsf{Shaves}(c, c)$$

$$\mathsf{KR}(c) \land \neg\mathsf{Shaves}(c, c)$$

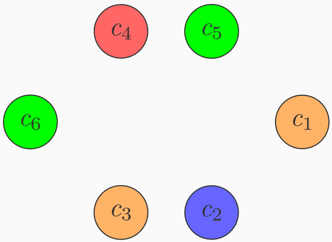
$$\mathsf{KR}(c) \land \mathsf{Shaves}(c, c)$$

1- Type Enumeration

An arrangement of 1-Types

$\neg\mathsf{KR}(\mathsf{c}) \land \neg\mathsf{Shaves}(\mathsf{c},\mathsf{c})$

$\neg\mathsf{KR}(\mathsf{c}) \land \mathsf{Shaves}(\mathsf{c},\mathsf{c})$



$\boldsymbol{k}:\quad k_1 = 1 \quad k_2 = 2 \quad k_3 = 1 \quad k_4 = 2$

$\mathsf{KR}(\mathsf{c}) \land \neg\mathsf{Shaves}(\mathsf{c},\mathsf{c})$

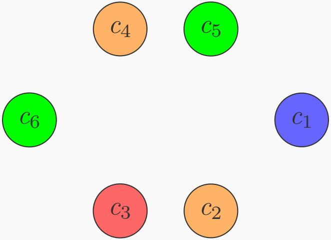
$\mathsf{KR}(\mathsf{c}) \land \mathsf{Shaves}(\mathsf{c},\mathsf{c})$

1- Type Enumeration

Another arrangement of 1-Types

$\neg$KR(c) $\wedge$ $\neg$Shaves(c, c)

$\neg$KR(c) $\wedge$ Shaves(c, c)



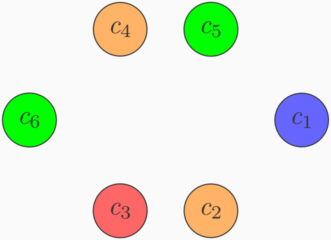
$\boldsymbol{k}:\quad k_1 = 1 \quad k_2 = 2 \quad k_3 = 1 \quad k_4 = 2$

KR(c) $\wedge$ $\neg$Shaves(c, c)

KR(c) $\wedge$ Shaves(c, c)

1-Type Enumeration

Counting for fixed 1-Type cardinalities



$$\boldsymbol{k}: \quad k_1 = 1 \quad k_2 = 2 \quad k_3 = 1 \quad k_4 = 2$$

$$\#\text{Similar Arrangements} = \binom{n}{k_1, k_2, k_3, k_4} = \frac{6!}{1!2!1!2!} = 180$$

FO^2 Language: 2-tables

We have an FO^2 language, with the following predicates:

- A unary predicate $\mathsf{KR}(x)$
- A binary predicate $\mathsf{Shaves}(x, y)$

We have the following set of binary properties also called 2-tables:

$$\mathsf{Shaves}(c, d) \land \mathsf{Shaves}(d, c)$$

$$\neg\mathsf{Shaves}(c, d) \land \mathsf{Shaves}(d, c)$$

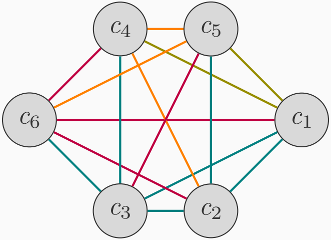
$$\mathsf{Shaves}(c, d) \land \neg\mathsf{Shaves}(d, c)$$

$$\neg\mathsf{Shaves}(c, d) \land \neg\mathsf{Shaves}(d, c)$$

2- Table Enumeration

An arrangement for 2-tables

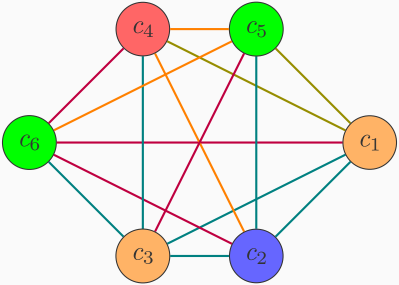
$\mathsf{Shaves}(c, d) \wedge \mathsf{Shaves}(d, c)$ $\neg\mathsf{Shaves}(c, d) \wedge \mathsf{Shaves}(d, c)$



$\mathsf{Shaves}(c, d) \wedge \neg\mathsf{Shaves}(d, c)$ $\neg\mathsf{Shaves}(c, d) \wedge \neg\mathsf{Shaves}(d, c)$

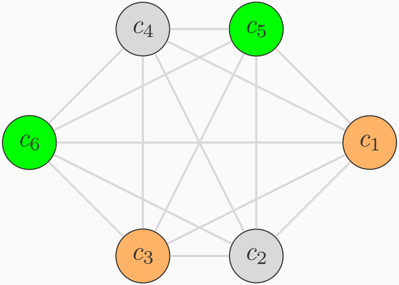
2- Table Enumeration given 1-Types

An arrangement for 2-tables given 1-types



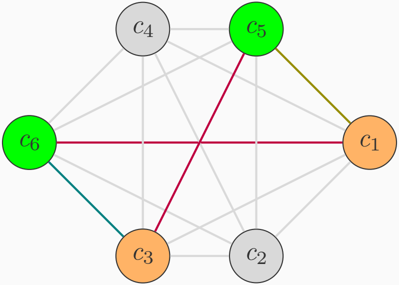
2- Table Enumeration given 1-Types

Pick a pair of 1-Types



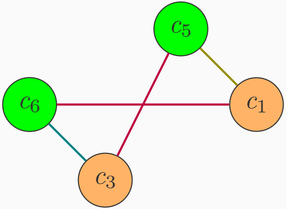
2- Table Enumeration given 1-Types

Picking a sub graph: Pick 2-Tables between them



2- Table Enumeration

Enumerating 2-tables given 1-types

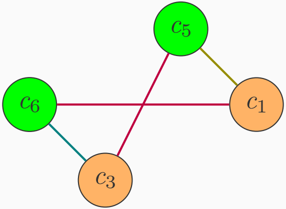


$k_2 = 2 \quad k_4 = 2$

$h_1 = 2 \quad h_2 = 1 \quad h_3 = 1 \quad h_4 = 0$

2- Table Enumeration

Enumerating 2-tables given 1-types



$$k_2 = 2 \quad k_4 = 2$$

$$h_1 = 2 \quad h_2 = 1 \quad h_3 = 1 \quad h_4 = 0$$

$$\binom{k_2 \times k_4}{h_1 \quad h_2 \quad h_3 \quad h_4} = \frac{(2 \times 2)!}{2!2!1!0!} = 6$$

Enumerating all models over 1-types and 2-tables

$$\sum_{\langle \vec{\boldsymbol{k}}, \vec{\boldsymbol{h}} \rangle} \binom{n}{k_1 \ldots k_u} \prod_{1 \leq i \leq j \leq u} \binom{k(i,j)}{h_1^{ij} \ldots h_b^{ij}}$$

$$\boldsymbol{k}(i,j) = \begin{cases} k_i k_j & \text{if } i \neq j \\ \frac{k_i(k_i-1)}{2} & \text{if } i = j \end{cases}$$

Adding Formulas : $\forall xy.\Phi(x, y)$

A formula $\forall xy.\Phi(x, y)$, allows some and disallows other 1-Type and 2-Table configuration. For Example:

$$\neg \mathsf{Shaves}(x, x)$$

$$\mathsf{Shaves}(x, y) \rightarrow \mathsf{Shaves}(y, x)$$

$$\mathsf{KR}(x) \land \mathsf{Shaves}(x, y) \rightarrow \mathsf{KR}(y)$$

Adding Formulas : $\forall xy.\Phi(x, y)$

$$\neg\mathsf{Shaves}(x, x)$$
$$\mathsf{Shaves}(x, y) \rightarrow \mathsf{Shaves}(y, x)$$
$$\mathsf{KR}(x) \wedge \mathsf{Shaves}(x, y) \rightarrow \mathsf{KR}(y)$$

Not allowed:

$$\neg\mathsf{KR}(c) \wedge \neg\mathsf{Sh}(c, c) \wedge \mathsf{KR}(d) \wedge \neg\mathsf{Sh}(d, d) \wedge \mathsf{Sh}(c, d) \wedge \mathsf{Sh}(d, c)$$

Allowed:

$$\mathsf{KR}(c) \wedge \neg\mathsf{Sh}(c, c) \wedge \mathsf{KR}(d) \wedge \neg\mathsf{Sh}(d, d) \wedge \mathsf{Sh}(c, d) \wedge \mathsf{Sh}(d, c)$$

FOMC in FO^2: $\forall x \forall y . \Phi(x, y)$

$$\mathrm{FOMC}(\Phi, n) =$$

$$\sum_{\langle \vec{\boldsymbol{k}}, \vec{\boldsymbol{h}} \rangle} \binom{n}{k_1, \dots, k_u} \prod_{1 \le i \le j \le u} \binom{\boldsymbol{k}(i,j)}{h_1^{ij}, \dots, h_b^{ij}} \prod_{1 \le v \le b} {n_{ijv}}^{h_v^{ij}}$$

Unary Properties

Constraints: Φ

Binary Properties

FOMC in FO^2: $\forall x \forall y. \Phi(x, y)$

$$\mathrm{FOMC}(\Phi, n) =$$

$$\sum_{\langle \vec{\boldsymbol{k}}, \vec{\boldsymbol{h}} \rangle} \binom{n}{k_1, \ldots, k_u} \prod_{1 \le i \le j \le u} \binom{\boldsymbol{k}(i, j)}{h_1^{ij}, \ldots, h_b^{ij}} \prod_{1 \le v \le b} {n_{ijv}}^{h_v^{ij}}$$

$$= \sum_{\langle \vec{\boldsymbol{k}}, \vec{\boldsymbol{h}} \rangle} \mathsf{FOMC}(\Phi, \langle \vec{\boldsymbol{k}}, \vec{\boldsymbol{h}} \rangle)$$

$$= \sum_{\vec{\boldsymbol{k}}} \mathsf{FOMC}(\Phi, \vec{\boldsymbol{k}})$$

Cardinality Constraints

$$\neg \mathsf{Shaves}(x, x)$$

$$\mathsf{Shaves}(x, y) \rightarrow \mathsf{Shaves}(y, x)$$

$$\mathsf{KR}(x) \wedge \mathsf{Shaves}(x, y) \rightarrow \mathsf{KR}(y)$$

$$|\mathsf{KR}(x)| = 2$$

Cardinality Constraints

$$\neg \mathsf{Shaves}(x, x)$$

$$\mathsf{Shaves}(x, y) \rightarrow \mathsf{Shaves}(y, x)$$

$$\mathsf{KR}(x) \wedge \mathsf{Shaves}(x, y) \rightarrow \mathsf{KR}(y)$$

$$|\mathsf{KR}(x)| = 2$$

k_1 : $\neg \mathsf{KR}(c) \wedge \neg \mathsf{Shaves}(c, c)$

k_2 : $\neg \mathsf{KR}(c) \wedge \mathsf{Shaves}(c, c)$

k_3 : $\mathsf{KR}(c) \wedge \neg \mathsf{Shaves}(c, c)$

k_4 : $\mathsf{KR}(c) \wedge \mathsf{Shaves}(c, c)$

Cardinality Constraints

$$\neg \mathsf{Shaves}(x, x)$$
$$\mathsf{Shaves}(x, y) \rightarrow \mathsf{Shaves}(y, x)$$
$$\mathsf{KR}(x) \wedge \mathsf{Shaves}(x, y) \rightarrow \mathsf{KR}(y)$$
$$|\mathsf{KR}(x)| = 2$$

$$k_1 : \neg \mathsf{KR}(c) \wedge \neg \mathsf{Shaves}(c, c)$$
$$k_2 : \neg \mathsf{KR}(c) \wedge \mathsf{Shaves}(c, c)$$
$$k_3 : \mathsf{KR}(c) \wedge \neg \mathsf{Shaves}(c, c)$$
$$k_4 : \mathsf{KR}(c) \wedge \mathsf{Shaves}(c, c)$$

$$\boldsymbol{k} : k_3 + k_4 = 2$$

Cardinality Constraints

Given an arbitrary cardinality constraint ρ:

$$\mathsf{FOMC}(\Phi \wedge \rho) := \sum_{\langle \boldsymbol{k}, \boldsymbol{h} \rangle \models \rho} \mathsf{FOMC}(\Phi, \langle \boldsymbol{k}, \boldsymbol{h} \rangle)$$

Existential Quantifiers [7]

Minimal Non-Trivial Example:

$$\Psi := \Phi \wedge \forall x \exists y. R(x, y)$$

[7] Guy Van den Broeck, Wannes Meert, Adnan Darwiche. KR 2014
Malhotra and Serrafini. AAAI 2022 (for this formulation)

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- FOMC(Φ) - $|\bigcup_{i \in [n]} A_i|$
- Use Principle of Inclusion Exclusion to calculate $|\bigcup_{i \in [n]} A_i|$

[7] Guy Van den Broeck, Wannes Meert, Adnan Darwiche. KR 2014
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Principle of Inclusion-Exclusion

- $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|$

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- $\bigcap_{j \in J} A_j$ is dependent on $|J|$ and not on $\{J\}$
- Let $\alpha_K = \left|\bigcap_{j \in J} A_j\right|$
- $\alpha_k = |\{\omega | \omega \models \Phi \wedge \forall xy. P(x) \rightarrow \neg R(x, y) \wedge |P| = k\}|$

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Recall,

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Recall,

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We wanted:

$$\text{FOMC}(\Phi) - |\bigcup_{i \in [n]} A_i|$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \alpha_k$$

$$\left|U \setminus \bigcup_{i=1}^{n} A_i\right| = \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \alpha_k$$

Existential Quantifier ≡PIE ≡WFOMC with negative weights

$$\text{WFOMC}(\Phi \wedge \forall x \exists y.R(x,y)) \equiv \text{WFOMC}(\Phi \wedge \forall xy.P(x) \rightarrow \neg R(x,y))$$

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Scott's Normal Form is equi-satisfiable and WFOMC preserving:

$$\Phi \bigwedge_i \forall x \exists y R_i(x,y)$$

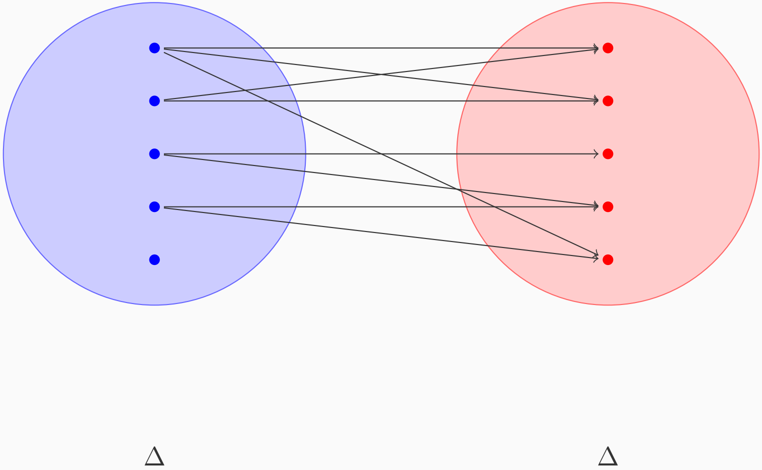
Counting Quantifiers [8]

Counting Quantifiers: $\forall x \exists^{=1} y . R(x, y)$

[8] Kuzelka. JAIR 2021. Functionality: Kuuisto, Lutz. LICS 2018

Counting Quantifiers: Expressing Functionality

$$R(x, y) : \Delta \to \Delta$$



Counting Quantifiers: Expressing Functionality

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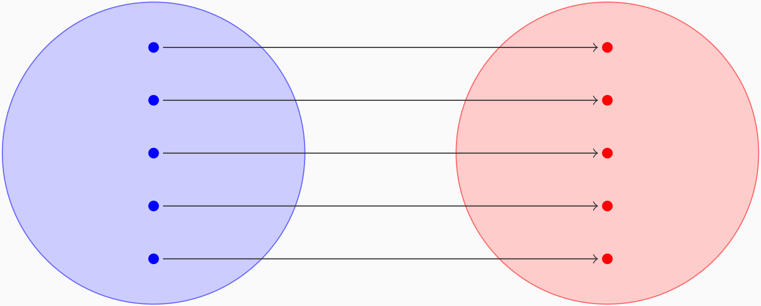

Δ Δ

Counting Quantifiers: Expressing Functionality

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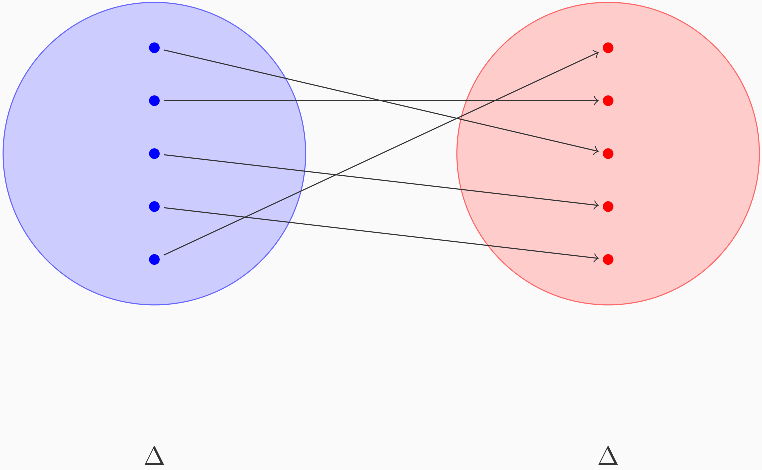

Counting Quantifiers: Expressing Functionality

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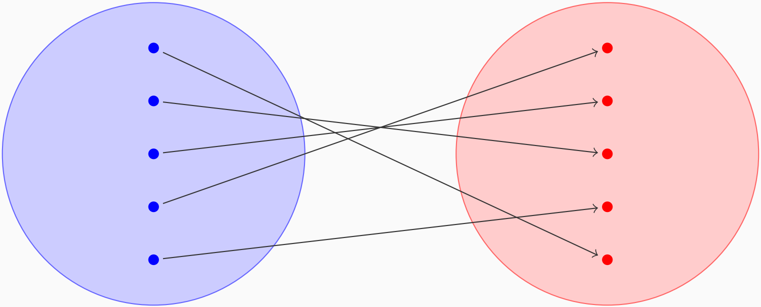

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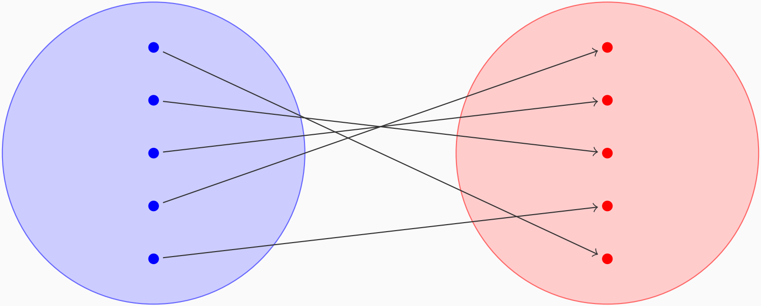

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Counting Quantifiers: Expressing Functionality

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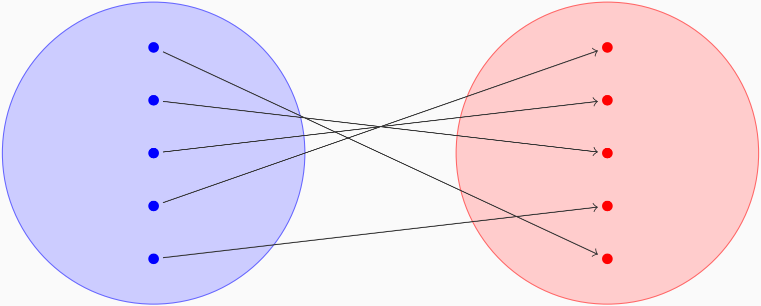

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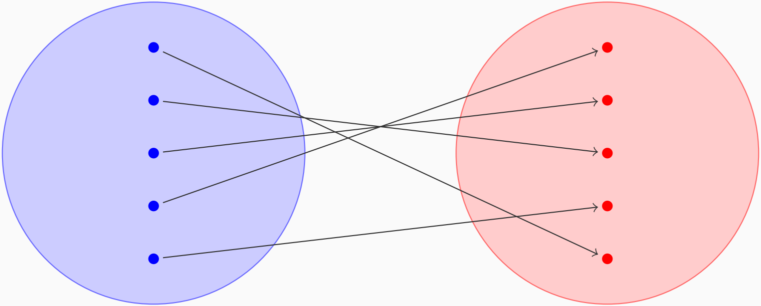


What is the cardinality of R in all functions ??

$$|\Delta|$$

$$\forall x.\exists^{=1} y.R(x, y) \equiv \forall x.\exists y.R(x, y) \wedge (|R| = |\Delta|)$$

Counting Quantifiers[9]

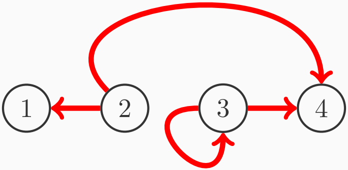
Let Φ be the following C^2 formula, then it can be written as:

$$\Phi_0 \wedge \bigwedge_{k=1}^{q} \forall x.(A_k(x) \leftrightarrow \exists^{=m_k} y.R_k(x, y))$$

where Φ_0 is a pure universal formula in FO^2.

[9] Kuzelka. JAIR 2021, Malhotra and Serafini. AAAI 2022 for this formulation

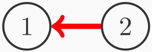
WFOMC beyond first-order logic [10]



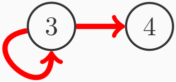
[10] Malhotra, Bizzaro and Serafini. 2024 (Under Review at Artificial Intelligence Journal)

Projections on sub-domain

Then, $\omega' = \omega \downarrow [2]$ and $\omega'' = \omega \downarrow [\bar{2}]$ are given respectively as



and



In how many ways can I merge them?

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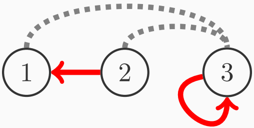

What is the number of ways to merge these partial interpretations?

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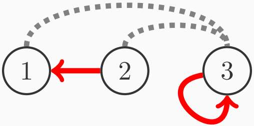

What is the number of ways to merge these partial interpretations?

$$2^4$$

In how many ways can I merge under constraints?

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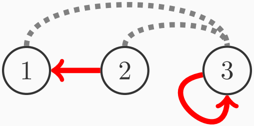

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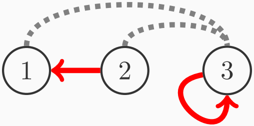

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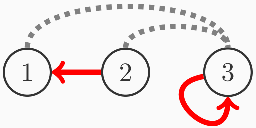

What is the number of ways to merge these partial interpretations? Such that there are only directed edges from $[2]$ to $[\bar{2}]$?

$$2^2$$

This can be captured by restricting the allowed 2-types between Δ and Δ'

Counting by splitting

How can you compute WFOMC of all the models on $[m]$ and $[\bar{m}]$, such that:

C1: $\omega \downarrow [m] \models axiom'$

C2: $\omega \downarrow [\bar{m}] \models axiom''$

C3: $\omega \models \forall x \in [m]\ \forall y \in [\bar{m}].\Theta(x, y)$

C4: $\omega \models \forall xy.\Phi(x, y)$

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C4: $\omega \models \forall xy.\Phi(x, y)$

- $\mathsf{W}(\omega')\mathsf{W}(\omega'') \prod_{i,j \in [u]} r_{ij}^{k_i' k_j''}$, where r_{ij} captures the constrains across them.
- $$\mathsf{WFOMC}(\Psi_{[m]}, \boldsymbol{k}) = \sum_{\substack{\boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{k} \\ |\boldsymbol{k}'| = m}} \mathsf{WFOMC}(\Psi', \boldsymbol{k}')\mathsf{WFOMC}(\Psi'', \boldsymbol{k}'') \prod_{i,j \in [u]} r_{ij}^{k_i' k_j''}$$

What can I do with this result?

	$axiom'$	$axiom''$	$\Theta(x, y)$
$DAG(R)$	$\forall xy.\neg R(x, y)$	$DAG(R)$	$\neg R(y, x)$
$Connected(R)$	$Connected(R)$	$\top$	$\neg R(x, y)$
$Forest(R)$	$Tree(R)$	$Forest(R)$	$\neg R(x, y)$

Table 1: A summary of results using counting by splitting

Open Problems

- Is there a some formal language that completely captures tractable WFOMC?

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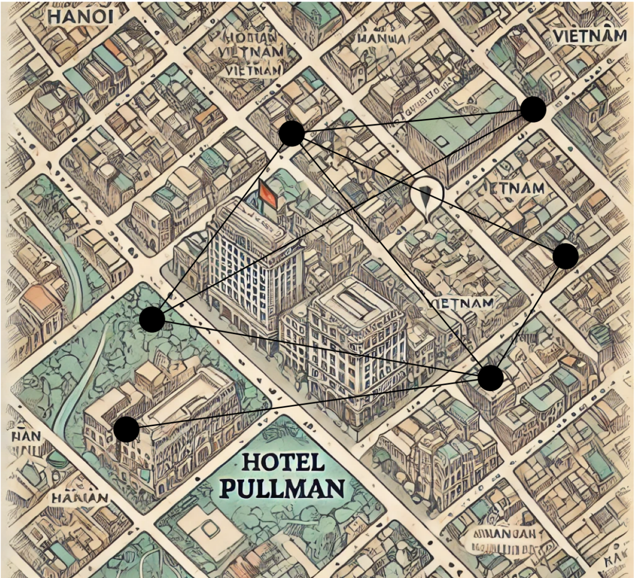
- Is there an FPRAS? (There is no PTAS (Jaeger 2014. TPLP))
- Evidence. Something is known about FPT (Broeck and Darwiche 2013. NeurIps). But not alot.
- Sampling – Amazing recent developments (Wang et al. LICS 2023)

Consistency

A Covid Outbreak in Pullman Hotel!



Back to my grim Example

A contact-tracing model:

$$w_1 : \texttt{Covid}(\texttt{x})$$
$$w_2 : \texttt{Covid}(\texttt{x}) \land \texttt{Contact}(\texttt{x}, \texttt{y}) \rightarrow \texttt{Covid}(\texttt{y})$$

$$\widehat{\mathbf{a}} = \underset{\mathbf{a}}{\operatorname{argmax}} \, P_{\Phi}^{(m)}(\omega) \tag{4}$$

A Covid Outbreak in Pullman Hotel!

What do we observe, and what do we learn?

How does one usually do parameter estimation?

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How does one usually do parameter estimation?

$$\widehat{\mathbf{a}} = \operatorname*{argmax}_{\mathbf{a}} P_{\Phi}^{(m)}(\omega) \tag{5}$$

What do we actually want to do?

$$\widehat{\mathbf{a}} = \operatorname*{argmax}_{\mathbf{a}} P_{\Phi}^{(n+m)} \downarrow [n](\omega)$$

where,

$$P^{(n)} \downarrow [m](\omega') = \sum_{\omega \in \Omega^{(n)} : \omega \downarrow [m] = \omega'} P^{(n)}(\omega)$$

Consistency of Learning and Inference

Consistency of Inference [Projectivity][11]: A model learnt over a domain of size n, basically says nothing quantitative about a domain of size $n+1$

$$P_{\boldsymbol{\theta}}^{(m)}(q) \neq P_{\boldsymbol{\theta}}^{(n)} \downarrow [m](q)$$

[11] Shalizi and Rinaldo. 2013. Jaeger and Shulte. StarAI Workshop 2018. Mittal, Bhardwaj, Gogate, Singla. AISTATS 2019.

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Consistency of Learning: Batching or stochastic gradient descent cannot work for MLNs ! as they donnot admit consistent parameter estimates

$$\underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, E_{\omega'}[\log P_{\boldsymbol{\theta}}^{(m)}(\omega')] \neq \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \log P_{\boldsymbol{\theta}}^{(n)}(\omega)$$

[11] Shalizi and Rinaldo. 2013. Jaeger and Shulte. StarAI Workshop 2018. Mittal, Bhardwaj, Gogate, Singla. AISTATS 2019.

Trying to get near Projectivity!

Can we improve the learning procedure to get near-projective models [12]

[12] Chen, Weitkamper, Malhotra. ECML 2024.
(For theory of RMP) Kuzelka, Wang, Davis and Schockaert. AAAI 2018.

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Can we improve the learning procedure to get near-projective models [12]

$$-\log P_{\Phi}^{(n+m)} \downarrow [n](\omega) \leq -\log P_{\Phi}^{(n)}(\omega) + \log \Delta \tag{6}$$

$$KL(P_{\Phi}^{(n+m)} \downarrow [n] || P_{\Phi}^{(n)}) \leq \log \Delta \tag{7}$$

[12] Chen, Weitkamper, Malhotra. ECML 2024.
(For theory of RMP) Kuzelka, Wang, Davis and Schockaert. AAAI 2018.

But what are Projective Models?

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Erdős–Rényi random graph:

$$\Pr(G(n+m,p)=G)=p^{|E|}(1-p)^{\binom{n+m}{2}-|E|}$$

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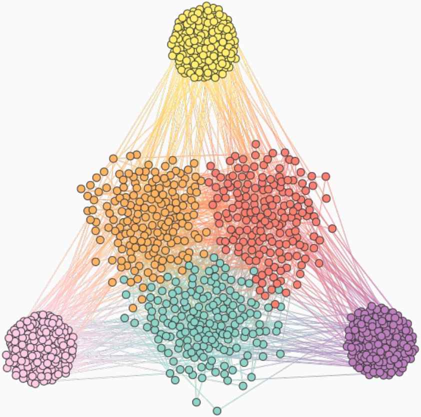
Erdős–Rényi random graph:

$$\Pr(G(n+m,p) = G) = p^{|E|}(1-p)^{\binom{n+m}{2}-|E|}$$

$$\begin{aligned} \Pr(G(n+m,p)\downarrow[n] = G') &= p^{|E'|}(1-p)^{\binom{n}{2}-|E'|} && (8)\\ &= \Pr(G(n,p) = G'). && (9)\end{aligned}$$

Can we express any projective models in MLNs?

"A Markov Logic Network in the Two Variable Fragment is projective if and only if it represents a stochastic block structure [13]"



[13] Malhotra and Serafini. ECML 2022

PROBABILITY PARAMETERS

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- w_{ijl} is the conditional probability of an arbitrary pair of domain constants to realise the lth 2-table, given they realise the i^{th} and the j^{th} 1-type.

$$w_{ijl} = P\big(l(x,y)|i(x) \land j(y)\big)$$

"Who Shaves Who in KR?"

Assuming **Shaves** to be irreflexive and symmetric, we have the following non-zero $\{p_i, w_{ijl}\}$

$$
\begin{aligned}
p_1 &= P(\mathsf{KR}(\mathsf{a})) \\
p_2 &= P(\neg\mathsf{KR}(\mathsf{a})) \\
w_{111} &= P(\mathsf{Shaves}(\mathsf{a},\mathsf{b})|\mathsf{KR}(\mathsf{a}),\mathsf{KR}(\mathsf{b})) \\
w_{112} &= P(\neg\mathsf{Shaves}(\mathsf{a},\mathsf{b})|\mathsf{KR}(\mathsf{a}),\mathsf{KR}(\mathsf{b})) \\
w_{121} &= P(\mathsf{Shaves}(\mathsf{a},\mathsf{b})|\mathsf{KR}(\mathsf{a}),\neg\mathsf{KR}(\mathsf{b})) \\
w_{122} &= P(\neg\mathsf{Shaves}(\mathsf{a},\mathsf{b})|\mathsf{KR}(\mathsf{a}),\neg\mathsf{KR}(\mathsf{b})) \\
w_{221} &= P(\mathsf{Shaves}(\mathsf{a},\mathsf{b})|\neg\mathsf{KR}(\mathsf{a}),\neg\mathsf{KR}(\mathsf{b})) \\
w_{222} &= P(\neg\mathsf{Shaves}(\mathsf{a},\mathsf{b})|\neg\mathsf{KR}(\mathsf{a}),\neg\mathsf{KR}(\mathsf{b}))
\end{aligned}
$$

The Relational Block Model

Relational Block Model :

$$P(\boldsymbol{X} = \boldsymbol{x}) := \prod_{q=1}^{n} p_{x_q} = \prod_{i=1}^{u} p_{x_i}^{k_i}$$

$$P(\boldsymbol{Y} = \boldsymbol{y} | \boldsymbol{X} = \boldsymbol{x}) := \prod_{1 \leq q < r \leq n} w_{x_q x_r y_{qr}}$$

$$= \prod_{1 \leq i \leq j \leq u} \prod_{1 \leq l \leq b} (w_{ijl})^{h_l^{ij}}$$

Any projective MLN in the two variable fragment reduces to an RBM

Malhotra and Serafini. ECML PKDD 2022

What is the most general class of projective models on graphs?

[14] László Lovász and Aldous-Hoover-Kallenberg

What is the most general class of projective models on graphs?

A **graphon**[14] is a symmetric, measurable function $W : [0,1] \times [0,1] \to [0,1]$ such that:

- $W(x,y) = W(y,x)$ for all $x, y \in [0,1]$ (symmetry),
- $W(x,y)$ gives the probability of an edge between points x and y.

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Sampling a graph from a graphon:

1. Sample n points $u_1, u_2, \ldots, u_n$ uniformly from $[0,1]$.
2. For each pair (i,j) with $i \neq j$, place an edge with probability $W(u_i, u_j)$.

[14] László Lovász and Aldous-Hoover-Kallenberg

What do all these models have in common?

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- Exchangeable Distributions:

$$P((X_{ij})) = P((X_{\sigma(i)\sigma(j)}))$$

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- X_{ij} are edges for a ER graph or a graphon
- X_{ij} are 2-types for FO^2
- Can be generalized to exchangeable random arrays like X_{ijl}

Aldous-Hoover-Kallenberg

Theorem 1 (Aldous-Hoover representation of jointly exchangeable matrices (Aldous, 1981; Hoover, 1979)). A random 2-array $(X_{ij})_{i,j \in \mathbb{N}}$ is jointly exchangeable if there exists a function $f : [0,1] \times [0,1]^2 \times [0,1] \to E$ such that

$$(X_{ij}) = (f(U, U_i, U_j, U_{ij})),$$

where $(U_i)_{i \in \mathbb{N}}$ and $(U_{ij})_{i,j>i \in \mathbb{N}}$ with $U_{ij} = U_{ji}$ are a sequence and matrix, respectively, of i.i.d. Uniform$[0,1]$ random variables.

Jaeger and Schulte. IJCAI 2020. For extension to Relational data.

Open Problems

- How to represent and learn expressive AHK models?

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- How to encode expert Knowledge into AHK models?

Thank You!