Weighted Model Counting in FO^2 with Cardinality Constraints and Counting Quantifiers

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Weighted First Order Model Counting

$$\mathrm{FOMC}(\Phi, n) = \sum_{\omega \in \Omega} \mathbb{1}(\omega \models \Phi)$$

$$\mathrm{WFOMC}(\Phi, n) = \sum_{\omega \in \Omega} \mathbb{1}(\omega \models \Phi) \times w(\omega)$$

Example:

$$\Phi = \forall \mathtt{xy}.\mathtt{Ax} \land \mathtt{Rxy} \rightarrow \mathtt{Ay}$$

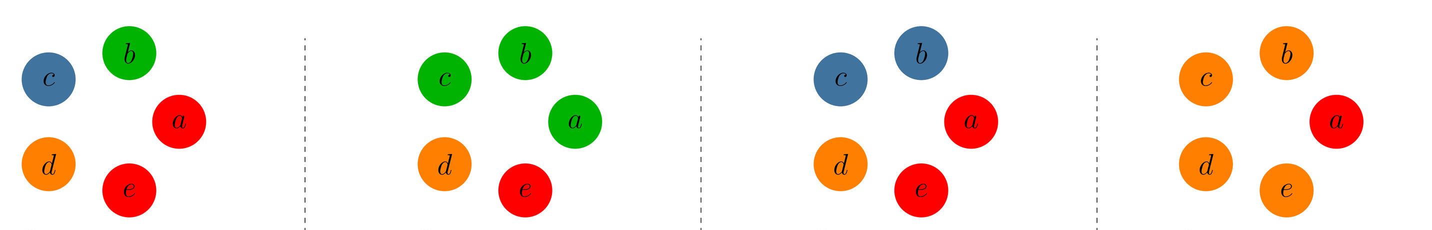
$\mathrm{FOMC}(\Phi, n)?$

Unary and Binary Properties in FO^2

Let us have a FOL language with a unary predicate $\mathtt{A}$ and a binary predicate $\mathtt{R}$. Then for any domain constant $\mathtt{c}$ exactly one of the following **unary** property is true:

$$\mathtt{Ac} \land \mathtt{Rcc} \quad | \quad \mathtt{Ac} \land \neg\mathtt{Rcc} \quad | \quad \neg\mathtt{Ac} \land \mathtt{Rcc} \quad | \quad \neg\mathtt{Ac} \land \neg\mathtt{Rcc} \tag{1}$$

For 5 domain elements some examples of **unary** configurations are given as follows:



$\binom{5}{2,1,1,1} = \frac{5!}{2!1!1!1!}$, $k_1 = 2$ $k_2 = 1$ $k_3 = 1$ $k_4 = 1$

$\binom{5}{1,3,0,1} = \frac{5!}{1!3!0!1!}$, $k_1 = 1$ $k_2 = 3$ $k_3 = 0$ $k_4 = 1$

$\binom{5}{2,0,2,1} = \frac{5!}{2!0!2!1!}$, $k_1 = 2$ $k_2 = 0$ $k_3 = 2$ $k_4 = 1$

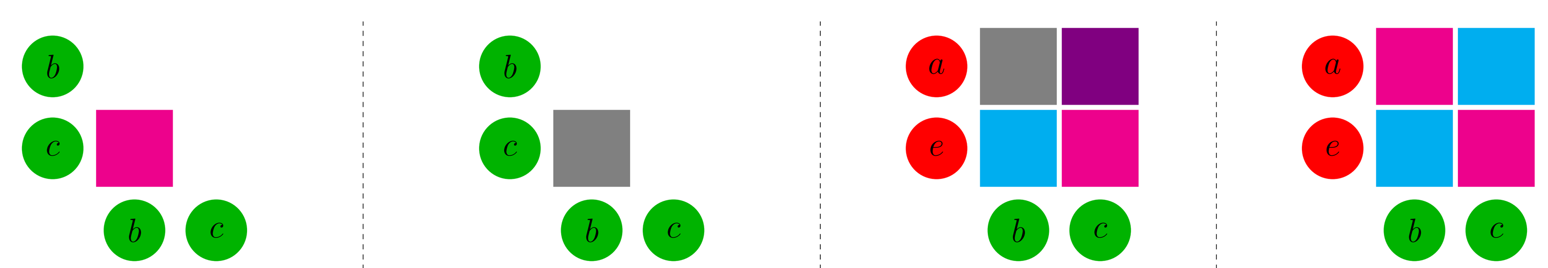
$\binom{5}{1,0,0,4} = \frac{5!}{1!0!0!4!}$, $k_1 = 2$ $k_2 = 1$ $k_3 = 1$ $k_4 = 1$

*In general, for a language with u **unary** properties over n domain elements, we have $\binom{n}{\vec{k}} = \frac{n!}{\prod_i k_i!}$ ways such that k_i constants realize the i^{th} property, where $\vec{k} = (k_1, ..., k_u)$.*

For any pair of domain constants $(\mathtt{c}, \mathtt{d})$, exactly one of the following **binary** property is true:

$$\mathtt{Rcd} \land \mathtt{Rdc} \quad | \quad \mathtt{Rcd} \land \neg\mathtt{Rdc} \quad | \quad \neg\mathtt{Rcd} \land \mathtt{Rdc} \quad | \quad \neg\mathtt{Rcd} \land \neg\mathtt{Rdc} \tag{2}$$

Given a unary configuration a e b c d of domain elements. Following are some possible realizations of binary properties by the pairs of domain elements.



$\binom{k_2 \cdot (k_2 - 1)/2}{0,0,1,0} = \frac{(1\times1)!}{0!1!0!0!}$, $h_1^{22} = 0$ $h_2^{22} = 0$ $h_3^{22} = 1$ $h_4^{22} = 0$

$\binom{k_2 \cdot (k_2 - 1)/2}{0,1,0,0} = \frac{(1\times1)!}{0!1!0!0!}$, $h_1^{22} = 0$ $h_2^{22} = 1$ $h_3^{22} = 0$ $h_4^{22} = 0$

$\binom{k_1 \cdot k_2}{1,1,1,1} = \frac{(2\times2)!}{1!1!1!1!}$, $h_1^{12} = 1$ $h_2^{12} = 1$ $h_3^{12} = 1$ $h_4^{12} = 1$

$\binom{k_1 \cdot k_2}{2,0,2,0} = \frac{(2\times2)!}{2!0!2!0!}$, $h_1^{12} = 2$ $h_2^{12} = 0$ $h_3^{12} = 2$ $h_4^{12} = 0$

*In general, for a language with b **binary** properties, given a configuration of unary properties by $\vec{k}$, then for any pair of unary properties i and j, we have $\binom{k(i,j)}{h_1^{ij} \dots h_b^{ij}}$ possible ways such that h_v^{ij} pairs of constants realize the v^{th} **binary** property, where*

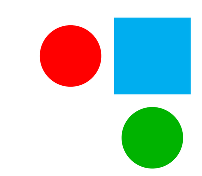
$$k(i,j) = \begin{cases} k_i \cdot (k_i - 1)/2 & i = j \\ k_i \cdot k_j & i \neq j \end{cases}$$

$\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}), n)$

Using arguments from the previous section we have that the number of interpretations such that k_i constants (say $\mathtt{c}$) realize the i^{th} **unary** property (denoted by $\mathtt{i}(\mathtt{c})$), and h_v^{ij} pairs of constants $(\mathtt{c}, \mathtt{d})$ such that $\mathtt{i}(\mathtt{c}) \land \mathtt{j}(\mathtt{d})$ and the pair $(\mathtt{c}, \mathtt{d})$ realizes the v^{th} **binary** property i.e. $\mathtt{i}(\mathtt{c}) \land \mathtt{j}(\mathtt{d}) \land \mathtt{v}(\mathtt{c}, \mathtt{d})$ is given by :

$$\binom{n}{\vec{k}} \prod_{1 \leq i \leq j \leq u} \binom{k(i,j)}{\vec{h^{ij}}} \tag{3}$$

$\omega \models \forall xy.\Phi(x, y)$ if and only if all the property configurations of each pair of domain constants in ω is allowed by the formula $\forall xy.\Phi(x,y)$. For example, $\forall \mathtt{xy}.\mathtt{Ax} \land \mathtt{Rxy} \rightarrow \mathtt{Ay}$ does not allow a pair of constants $(\mathtt{c}, \mathtt{d})$ such that $\mathtt{Ac} \land \mathtt{Rcc} \land \neg\mathtt{Ad} \land \neg\mathtt{Rdd} \land \mathtt{Rcd} \land \mathtt{Rdc}$ i.e. the following sub-structure is never allowed:



Hence, we introduce an indicator variable n_{ijv} for each configuration $\mathtt{i}(\mathtt{c}) \land \mathtt{j}(\mathtt{d}) \land \mathtt{v}(\mathtt{c}, \mathtt{d})$ which is 1 if:

$$\mathtt{i}(\mathtt{x}) \land \mathtt{j}(\mathtt{y}) \land \mathtt{v}(\mathtt{x}, \mathtt{y}) \models \Phi(\mathtt{x}, \mathtt{x}) \land \Phi(\mathtt{x}, \mathtt{y}) \land \Phi(\mathtt{y}, \mathtt{x}) \land \Phi(\mathtt{y}, \mathtt{y})$$

and 0 otherwise.

Hence, given a configuration represented by $\vec{k}$ and $\{\vec{h^{ij}}\}_{ij}$ we have the following possible realizations:

$$F(\vec{k}, \vec{h}, \{n_{ijv}\}) = \binom{n}{\vec{k}} \prod_{1 \leq i \leq j \leq u} \binom{k(i,j)}{\vec{h^{ij}}} \prod_{0 \leq v \leq b} (n_{ijv})^{h_v^{ij}} \tag{4}$$

Hence,

$$\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}), n) = \sum_{\vec{k}, \vec{h}} F(\vec{k}, \vec{h}, \{n_{ijv}\}) \tag{5}$$

Cardinality Constraints

Cardinality Constraints are constraints on the number of times a certain predicate is true in a given FOL interpretation.

Example:

$$\Phi := (\forall \mathtt{xy}.\mathtt{Ax} \land \mathtt{Rxy} \rightarrow \mathtt{Ay}) \land (|\mathtt{A}| = \mathtt{m})$$

Counting with a Cardinality Constraint ρ can be done by simply allowing cardinality configurations of the properties, which agree with the cardinality constraint.

$$\mathrm{FOMC}(\Phi \land \rho, n) = \sum_{\rho \models \vec{k}, \vec{h}} F(\vec{k}, \vec{h}, \{n_{ijv}\}) \tag{6}$$

In the above example, we can obtain the cardinality constraint by simply defining

$$\rho := k_1 + k_2 = m$$

Principle of Inclusion Exclusion

- Let Ω be a set of objects
- $\mathcal{S} = \{S_1, \dots, S_m\}$ be a set of properties of Ω
- e_0 : The count of objects with NONE of the properties in $\mathcal{S}$
- Let $Q \subseteq S$, then N_Q is the count of objects with AT LEAST the properties in Q

We define,

$$s_l = \sum_{|Q|=l} N_Q \tag{7}$$

Then the following relation holds:

$$e_0 = \sum_{l=0}^{m} (-1)^l s_l \tag{8}$$

Existential Quantifiers (Special Case)

$\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x} \exists \mathtt{y}.\mathtt{Rxy}, n)?$

$$\Omega = \{\omega : \omega \models \forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y})\} \tag{9}$$

$$S_c = \{\omega : \omega \models \forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{y}.\neg\mathtt{Rcy}\} \tag{10}$$

$$s_l = \mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \mathtt{Px} \rightarrow \neg\mathtt{Rxy} \land (|\mathtt{P}| = l)) \tag{11}$$

$$e_0 = \mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x} \exists \mathtt{y}.\mathtt{Rxy}) \tag{12}$$

Counting Quantifiers (Special Case)

$\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.(\mathtt{A}(\mathtt{x}) \leftrightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy}), n)?$

STEP 1: FOMC for :

$$\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.((\mathtt{Ax} \lor \mathtt{Bx}) \rightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy}) \land \forall \mathtt{x}.(\mathtt{Bx} \rightarrow \neg\mathtt{Ax}) \tag{13}$$

which is equal to FOMC for:

$$\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.((\mathtt{Ax} \lor \mathtt{Bx}) \rightarrow \exists \mathtt{y}.\mathtt{Rxy}) \tag{14}$$

$$\land \forall \mathtt{x}.(\mathtt{Bx} \rightarrow \neg\mathtt{Ax}) \tag{15}$$

$$\land \forall \mathtt{xy}.\mathtt{Mxy} \leftrightarrow ((\mathtt{Ax} \lor \mathtt{Bx}) \land \mathtt{Rxy}) \tag{16}$$

$$\land |\mathtt{M}| = |\mathtt{A}| + |\mathtt{B}| \tag{17}$$

STEP 2: Inclusion Exclusion:

KEY IDEA: Let $S_c = \{\omega : \omega \models \neg\mathtt{Ac} \land \exists^{=1}\mathtt{y}.\mathtt{Rcy}\}$ Clearly, we want the count of models ω such that $\omega \notin S_c$ for any c i.e.

$$e_0 = \mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.(\mathtt{Ax} \leftrightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy}))$$

$$s_l = \mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.((\mathtt{Ax} \lor \mathtt{Bx}) \rightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy})) \land \forall \mathtt{x}.(\mathtt{Ax} \rightarrow \neg\mathtt{Bx}) \land (|\mathtt{B}| = l)) \tag{18}$$

Weighted Model Counting

FOMC can be converted to WFOMC by just adding a multiplicative factor $w(\vec{k}, \vec{h})$ to every occurrence of $F(\vec{k}, \vec{h}, \{n_{ijv}\})$ in any counting formula:

$$(\vec{k}, \vec{h}) \mapsto w(\vec{k}, \vec{h}) \in \mathbb{R}^+$$

$w(\vec{k}, \vec{h})$ is a strictly more expressive weight function than symmetric weight functions.

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