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# Weighted Model Counting in the Two Variable Fragments

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# Probabilistic Inference $\leftrightarrow$ Weighted Model Counting

- Weight Function $w$ over a set of worlds $\Omega$:

$$w : \Omega \to \mathbb{R}^+ \tag{1}$$

- Probability of a world $\omega \in \Omega$:

$$P(\omega) = \frac{w(\omega)}{\sum_{\omega' \in \Omega} w(\omega')} \tag{2}$$

- Probability of a logical formula $\Phi$:

$$P(\Phi) = \frac{\sum_{\omega \models \Phi} w(\omega)}{\sum_{\omega' \in \Omega} w(\omega')} \tag{3}$$

# Weighted First Order Model Counting

$$\mathrm{WFOMC}(\Phi, n) = \sum_{\omega \models \Phi} w(\omega) \tag{4}$$

- $\Phi$ is a **First Order Logic** formula, $n$ is the domain cardinality
- $w$ is independent of the domain elements

**GOAL**: Identifying **Analytical formulas** for **PTIME** WFOMC

# FOMC in $FO^2$

# $FO^2$: An Example

**Example: Undirected Graphs**

All simple undirected graphs can be interpreted as models of the following $FO^2$ formula:

$$\Phi = \forall xy. \neg R(x,x) \land (R(x,y) \to R(y,x)) \tag{5}$$

where the domain $\Delta$ is the set of nodes.

**Problem:** How many undirected graphs exist over $n$ nodes?

# $\mathrm{FOMC}(\forall xy.\Phi(x,y), n)$: A First Look

Unary Properties Binary Properties Constraints: $\Phi$

$$\mathrm{FOMC}(\forall xy.\Phi(x,y), n) = \sum_{\vec{k},\vec{h}} \binom{n}{\vec{k}} \prod_{1\le i\le j\le u} \binom{\boldsymbol{k}(i,j)}{\vec{\boldsymbol{h}}^{ij}} \prod_{1\le v\le b} {n_{ijv}}^{h_v^{ij}} \tag{6}$$

$$\boldsymbol{k}(i,j) = \begin{cases} \frac{k_i(k_i-1)}{2} & \text{if } i=j \\ k_i k_j & \text{otherwise} \end{cases} \tag{7}$$

# 1-types : "Unary Properties"

A **1-type** $\alpha(x)$ is a conjunction of maximally consistent set of literals containing **only the variable** $x$

**Example:**

In an FO$^2$ language consisting of only one binary predicate $R$ and a unary predicate $A$, the 1-types are given as follows:

$$\alpha_1(x) : \neg A(x) \land \neg R(x,x) \qquad \alpha_2(x) : \neg A(x) \land R(x,x)$$

$$\alpha_3(x) : A(x) \land \neg R(x,x) \qquad \alpha_4(x) : A(x) \land R(x,x)$$

$$\alpha_5(x) : \neg A(x) \land \neg R(x,x) \qquad \alpha_6(x) : \neg A(x) \land R(x,x)$$

$$\alpha_7(x) : A(x) \land \neg R(x,x) \qquad \alpha_8(x) : A(x) \land R(x,x)$$

# 1-types enumeration

**Key Idea**: **A given constant realizes exactly one 1-type**

## 1-Type Enumeration

Given a complete set of 1-types $\{\alpha_i\}_{i=1}^{u}$, then the number of ways of assigning 1-types to $n$ domain constants such that we have $k_i$ domain constants of type $\alpha_i$ is given as:

$$\binom{n}{\vec{\boldsymbol{k}}} = \frac{n!}{k_1! \times ... \times k_i! \times ... \times k_u!} \tag{8}$$

# 2-tables: Binary Properties

A **2-table** is a conjunction of maximally consistent literals with **exactly two variables** $\{x, y\}$

**Example: 2-Tables**

In an $FO^2$ language $\mathcal{L}$ with only one binary predicate $R$,the 2-tables are given as :

$$\beta_1(x, y) : \neg R(x, y) \land \neg R(y, x)$$
$$\beta_2(x, y) : \neg R(x, y) \land R(y, x)$$
$$\beta_3(x, y) : R(x, y) \land \neg R(y, x)$$
$$\beta_4(x, y) : R(x, y) \land R(y, x)$$

# 2-table enumeration

**Key Idea: A given pair of domain constants realize exactly one 2-table**

## Binary Property Enumeration

Given a complete set of 2-tables $\{\beta_v\}_{v=1}^{b}$, then the number of models $\omega$ such that :

$$k_i = |\{c : \omega \models \alpha_i(c)\}| \qquad k_j = |\{c : \omega \models \alpha_j(c)\}|$$

where $i \neq j$

Then the number of ways such that we have $h_v^{ij}$ pair of constants $(c, d)$ such that:

$$\alpha_i(c) \wedge \alpha_j(d) \wedge \beta_v(c, d)$$

is given as:

$$\binom{k_i k_j}{\vec{\boldsymbol{h}}^{ij}} = \frac{(k_i \times k_j)!}{h_1^{ij}! \times \ldots \times h_v^{ij}! \times \ldots \times h_b^{ij}!} \tag{9}$$

# Enumerating all models over 1-types and 2-tables

$$\sum_{\vec{k},\vec{h}} \binom{n}{k_1 \ldots k_u} \prod_{1 \le i \le j \le u} \binom{k(i,j)}{h_1^{ij} \ldots h_b^{ij}}$$

$$\boldsymbol{k}(i,j) = \begin{cases} \frac{k_i(k_i-1)}{2} & \textbf{if } i = j \\ k_i k_j & \textbf{otherwise} \end{cases}$$

# Adding Formulas : $\forall xy.\Phi(x, y)$

Given $\forall xy.\Phi(x, y)$

$n_{ijv}$ is 1 if :

$$\alpha_i(x) \land \alpha_j(y) \land \beta_v(x, y) \models \Phi(x, x) \land \Phi(x, y) \land \Phi(y, x) \land \Phi(y, x) \tag{10}$$

and 0 otherwise.

**$n_{ijv}$ are the indicator variables**

An interpretation $\omega \models \forall xy.\Phi(x, y)$ if:

$$\forall (c \neq d) : (\omega \models \alpha_i(c) \land \alpha_j(d) \land \beta_v(c, d)) \leftrightarrow (n_{ijv} = 1) \tag{11}$$

## FOMC in FO$^2$: $\forall x \forall y . \Phi(x, y)$

$$\mathrm{FOMC}(\Phi, n) =$$

$$\sum_{\vec{\boldsymbol{k}},\vec{\boldsymbol{h}}} \binom{n}{k_1, \ldots, k_u} \prod_{1 \le i \le j \le u} \binom{\boldsymbol{k}(i,j)}{h_1^{ij}, \ldots, h_b^{ij}} \prod_{1 \le v \le b} {n_{ijv}}^{h_v^{ij}}$$

**Unary Properties**

**Constraints:** $\Phi$

**Binary Properties**

# Cardinality Constraints

# Cardinality Constraints

Cardinality constraints are arithematic constraints on the number of times a certain predicate is true in a given model

**Example: Simple labelled Graphs with $m$ Edges**

$$\forall xy.\Phi(x, y) = \forall xy.(\neg R(x, x)) \land (R(x, y) \to R(y, x)) \land (|R| = 2m)$$

# Counting with Cardinality Constraints

**Key Idea**: The $\vec{k}$ and $\vec{h^{ij}}$ contain all the cardinality information

**Example: Simple labelled Undirected Graphs with $m$ Edges**

$\forall xy.\Phi(x, y) = \forall xy.(\neg R(x, x)) \wedge (R(x, y) \rightarrow R(y, x)) \wedge (|R| = 2m)$

$$\mathrm{FOMC} = \sum_{\vec{\boldsymbol{k}}, \vec{\boldsymbol{h}} \models \rho} \binom{n}{k_1, k_2} \prod_{1 \leq i \leq j \leq 2} \binom{\boldsymbol{k}(i, j)}{h_1^{ij}, \ldots, h_4^{ij}} \prod_{1 \leq v \leq 4} n_{ijv}^{h_v^{ij}}$$

$\rho$ is $k_2 + \sum_{i \leq j}(h_2^{ij} + h_3^{ij} + 2h_4^{ij}) = 2m$

# Existential Quantifiers

# Existential Quantifiers

Scott's Normal Form:

$$\forall xy.\Phi(x, y) \wedge \bigwedge_{i=1}^{m} \forall x \exists y.R_i(x, y) \tag{12}$$

A Special Case:

$$\forall xy.\Phi(x, y) \wedge \forall x \exists y.R(x, y) \tag{13}$$

# Principle of Inclusion Exclusion

- Let $\Omega$ be a set of objects
- $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a set of properties of $\Omega$
- $e_0$ : The count of objects with **NONE** of the properties in $S$
- Let $Q \subseteq S$, then $N_Q$ is the count of objects with **AT LEAST** the properties in $Q$

We define,

$$s_l = \sum_{|\mathcal{Q}|=l} N_Q \tag{14}$$

Then the following relation holds:

$$e_0 = \sum_{l=0}^{m} (-1)^l s_l \tag{15}$$

# FOMC Existential Quantifiers (Special case)

$$\forall xy.\Phi(x,y) \wedge \forall x \exists y.R(x,y)$$

$$\Omega = \{\omega : \omega \models \forall xy.\Phi(x,y)\} \tag{16}$$

$$S_c = \{\omega : \omega \models \forall xy.\Phi(x,y) \wedge \forall y.\neg R(c,y)\} \tag{17}$$

$$e_0 = \mathrm{FOMC}(\forall xy.\Phi(x,y) \wedge \forall x \exists y.R(x,y)) \tag{18}$$

$$s_l = \mathrm{FOMC}(\forall xy.\Phi(x,y) \wedge P(x) \rightarrow \neg R(x,y) \wedge (|P| = l)) \tag{19}$$

From principle of inclusion-exclusion:

$$e_0 = \sum_{l=1}^{n} (-1)^l s_l \tag{20}$$

# Counting Quantifiers ($C^2$)

# Counting Quantifiers

Counting Quantifiers : $\exists^{=k}, \exists^{\leq k}$ and $\exists^{\geq k}$

**Example: Every Dog has at-most 4 legs**

$$\forall x.Dog(x) \rightarrow \exists^{\leq 4} x.Legs(x)$$

# Counting Quantifiers: A simple reduction

$$\forall xy.\Phi(x, y) \wedge \bigwedge_{k=1}^{q} \forall x.(A_k(x) \leftrightarrow \exists^{=m_k} y.\Psi_k(x, y)) \tag{21}$$

Special case

$$\forall xy.\Phi(x, y) \wedge \forall x.(A(x) \leftrightarrow \exists^{=1} y.R(x, y)) \tag{22}$$

# Counting Quantifiers: Step 1

$$\forall xy.\Phi(x,y) \wedge \forall x.(A(x) \vee B(x) \rightarrow \exists^{=1} y.R(x,y)) \wedge \forall x.(A(x) \rightarrow \neg B(x)) \tag{23}$$

# Counting Quantifiers: Step 1 (FOMC preserving reduction)

$$\forall xy.\Phi(x, y) \land \forall x.((A(x) \lor B(x)) \rightarrow \exists^{=1} y.R(x, y)) \land \forall x.(A(x) \rightarrow \neg B(x)) \tag{19}$$

Has the same FOMC as the following formula:

$$\forall xy.\Phi(x, y) \land \forall x.((A(x) \lor B(x)) \rightarrow \exists y.R(x, y)) \tag{24}$$

$$\land \forall x.(A(x) \rightarrow \neg B(x)) \tag{25}$$

$$\land \forall xy.M(x, y) \leftrightarrow ((A(x) \lor B(x)) \land R(x, y)) \tag{26}$$

$$\land |M| = |A| + |B| \tag{27}$$

# Counting Quantifiers: Step 2 (Inclusion Exclusion)

**KEY IDEA:** Let $S_c = \{\omega : \omega \models \neg A(c) \wedge \exists^{=1} y.R(c, y)\}$ Clearly, we want the count of models $\omega$ such that $\omega \notin S_c$ for any $c$ i.e.

$$e_0 = \mathrm{FOMC}(\forall xy.\Phi(x, y) \wedge \forall x.(A(x) \leftrightarrow \exists^{=1} y.R(x, y)))$$

$$\begin{aligned} s_l = \mathrm{FOMC}(\forall xy.\Phi(x, y) \wedge \forall x.((A(x) \vee B(x)) \rightarrow \exists^{=1} y.R(x, y)) \\ \wedge \forall x.(A(x) \rightarrow \neg B(x)) \wedge (|B| = l)) \end{aligned} \tag{28}$$

$$e_0 = \sum_{l=1}^{n} (-1)^l s_l \tag{29}$$

# Weighted Model Counting

# Weighted Model Counting

$$\text{FOMC}(\forall xy.\Phi(x,y), n) = \sum_{\vec{k},\vec{h}} \binom{n}{k_1, \ldots, k_u} \prod_{1 \le i \le j \le u} \binom{\boldsymbol{k}(i,j)}{h_1^{ij}, \ldots, h_b^{ij}} \prod_{1 \le v \le b} {n_{ijv}}^{h_v^{ij}}$$

$$= \sum_{\vec{k},\vec{h}} F(\vec{\boldsymbol{k}}, \vec{\boldsymbol{h}}, \{n_{ijv}\})$$

For WFOMC, we introduce $w : (\vec{\boldsymbol{k}}, \vec{\boldsymbol{h}}) \to \mathbb{R}^+$

$$\text{WFOMC}(\Phi, n) = \sum_{\vec{k},\vec{h}} w(\vec{\boldsymbol{k}}, \vec{\boldsymbol{h}}) F(\vec{\boldsymbol{k}}, \vec{\boldsymbol{h}}, \{n_{ijv}\})$$

## Expressivity of the weight functions $w(\vec{k}, \vec{h})$

$$\mathrm{WFOMC}(\Phi, n) = \sum_{\vec{k}, \vec{h}} w(\vec{k}, \vec{h}) F(\vec{k}, \vec{h}, \{n_{ijv}\})$$

**The weight functions $w(k, h)$ are a strictly more expressive class of weight functions than symmetric weight functions**

**Thank You !**